Qualitative properties of solutions of NLS on graphs

Nonlinear Quantum Graphs (Institut de Mathématiques de Toulouse)

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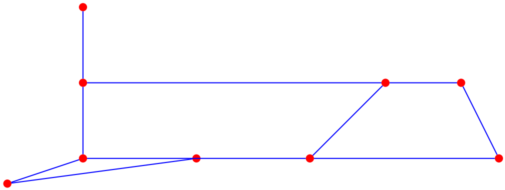
Compact metric graphs

A compact metric graph is made of a finite number of vertices and of finite length edges joining the vertices.



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where the symbol $e \succ \mathrm{v}$ means that the sum ranges over all edges of vertex v and where $\frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{v})$ is the outgoing derivative of u at v (*Kirchhoff's condition*).

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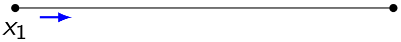
Kirchhoff's condition: degree one nodes



$$\lim_{t \xrightarrow[t>0]{} 0} \frac{u(x_1 + t) - u(x_1)}{t} = 0$$

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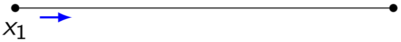
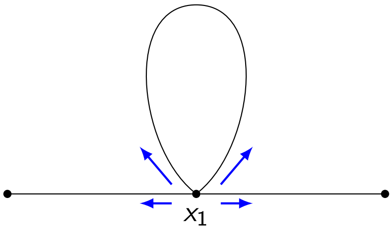


$$\lim_{\substack{t \to 0 \\ t>0}} \frac{u(x_1 + t) - u(x_1)}{t} = 0$$

In other words, the derivative of u at x_1 vanishes: this is the usual Neumann condition.

Kirchhoff's condition in general: outgoing derivatives



$$\sum_{e \succ \mathrm{v}} \frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{v}) = 0$$

Variational formulation and functional spaces

Sobolev spaces:

$$H^1(\mathcal{G}) := \left\{ u : \mathcal{G} \to \mathbb{R} \mid u \text{ is continuous}, u, u' \in L^2(\mathcal{G}) \right\},$$
$$H^1_Z(\mathcal{G}) := \left\{ u \in H^1(\mathcal{G}) \mid u(\mathrm{v}) = 0 \text{ for all } \mathrm{v} \in Z \right\}.$$

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Solutions of (NLS) correspond to critical points of

$$\frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{a}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})}.$$

over H^1_Z.

Idea: $p \to 2$

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which is *linear*.

Hope: obtain more information in the regime $p \approx 2$, by studying the *spectral* properties of the problem.

The eigenvalue problem

We denote by $(\lambda_k)_{k\geq 1}$ the sequence of eigenvalues of the problem

$$\begin{cases} -u'' + au = \lambda u & \text{on every edge of } \mathcal{G}, \\ u \text{ is continuous} & \text{on } \mathcal{G}, \\ \sum_{e \succ \mathrm{v}} u_e'(\mathrm{v}) = 0 & \text{for every } \mathrm{v} \in \mathbb{V} \setminus Z, \\ u(\mathrm{v}) = 0 & \text{for every } \mathrm{v} \in Z. \end{cases} \qquad (\mathcal{P}_2)$$

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E_k: eigenspace associated to λ_k.

Goal

For every positive integer k and $p > 2$, we want to relate solutions of the nonlinear problem

$$\begin{cases} -\tilde{u}'' + a\tilde{u} = |\tilde{u}|^{p-2}\tilde{u} & \text{on every edge of } \mathcal{G}, \\ \tilde{u} \text{ is continuous} & \text{on } \mathcal{G}, \\ \sum_{e \succ \mathrm{v}} \tilde{u}_e'(\mathrm{v}) = 0 & \text{for every } \mathrm{v} \in \mathbb{V} \setminus Z, \\ \tilde{u}(\mathrm{v}) = 0 & \text{for every } \mathrm{v} \in Z, \end{cases}$$

to the eigenfunctions of the eigenvalue problem $(\mathcal{P}_2)$ with eigenvalue λ_k.

A rescaling

In order to better understand the behaviour of the solutions as $p \to 2$, we consider the new variable $u = \lambda_k^{-1/(p-2)} \tilde{u}$.

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The rescaling will allow sequences of solutions of $(\mathcal{P}_{p,k})$ (with variable p) to be bounded in H^1 when $p \to 2$.

The reduced problem when $p \approx 2$

Let $(u_{p_n})_n$ be a sequence of solutions to $(\mathcal{P}_{p_n,k})$, $(p_n)_n \subseteq]2, +\infty[$, $p_n \to 2$.

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Question

What can we say about u_**?*

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Let $\varphi \in H^1_Z(\mathcal{G})$. Using φ as a test function in $(\mathcal{P}_{p_n,k})$, we get

$$\int_{\mathcal{G}} (u_{p_n}' \varphi' + a u_{p_n} \varphi) \, \mathrm{d}x = \lambda_k \int_{\mathcal{G}} |u_{p_n}|^{p_n - 2} u_{p_n} \varphi \, \mathrm{d}x.$$

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Taking the limit $n \to \infty$ leads to (since $p_n \to 2$)

$$\int_{\mathcal{G}} (u_*' \varphi' + a u_* \varphi) \, \mathrm{d}x = \lambda_k \int_{\mathcal{G}} u_* \varphi \, \mathrm{d}x.$$

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Question

Is that all we can say about u_*?

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Let us use specifically $\psi \in E_k$ as a test function in $(\mathcal{P}_{p_n,k})$. We obtain

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Thus,

$$\int_{\mathcal{G}} \left(|u_{p_n}|^{p_n - 2} - 1 \right) u_{p_n} \psi \, \mathrm{d}x = 0.$$

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We divide by $p_n - 2$:

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Definition

A function $u_* \in E_k$ is a **solution of the reduced problem on** E_k if and only if

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for all $\psi \in E_k$.

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Given a sequence $(u_{p_n})_n$, $p_n \to 2$ converging weakly to $u_* \in H^1_Z$, we have seen that necessarily:

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Question

Given a solution of the reduced problem $u_* \in E_k$, can one find solutions of $(\mathcal{P}_{p,k})$ close to u_* for $p \approx 2$? Can one detect when there is only one solution close to u_* for a given $p \approx 2$?

Lyapunov-Schmidt reduction

Functional space with extra regularity:

$$H := \left\{ u \in H^1_Z \mid u \text{ is } H^2 \text{ in each edge}, u \text{ satisfies Kirchhoff's conditions} \right\}.$$

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We fix $k \geq 1$ and we define the map

$$F : \begin{cases} [2, +\infty[\times H & \to L^2(\mathcal{G}), \\ (p, u) & \mapsto -u'' + au - \lambda_k |u|^{p-2} u. \end{cases}$$

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$$F(p, u) = 0 \iff u \text{ solves } (\mathcal{P}_{p,k}).$$

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we obtain good invertibility properties on $E_k^\perp$ and we are then reduced to a finite dimensional problem on E_k.

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To perform the Lyapunov-Schmidt reduction around u_*, we will need

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to be $\mathcal{C}^2$ in u in the neighborhood of $(2, u_*)$.

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Remark: if $u \in E_k$, then

$$\left(u \in S \right) \iff u \text{ does not vanish identically on edge of } \mathcal{G}.$$

Nondegenerate solutions of the reduced problem

Definition

A solution $u_* \in E_k \cap S$ of the reduced problem on E_k is **nondegenerate** if and only if the map

$$E_k \to E_k : \psi \mapsto P_{E_k}\Big((1 + \ln|u_*|)\psi\Big)$$

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Remark: nondegeneracy always holds if $\dim E_k = 1$.

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Let $k \geq 1$ be an integer and let $u_* \in E_k \cap S$.

1. ***non-existence**: If u_* is not a solution of the reduced problem, then there exists a neighbourhood U of $(2, u_*)$ in $[2, +\infty[\times H$ so that problem $(\mathcal{P}_{p,k})$ has no solution in U with $p > 2$;*
2. ***existence, uniqueness and non-degeneracy**: If u_* is a nondegenerate solution of the reduced problem, then there exists a neighbourhood U of $(2, u_*)$ in $[2, +\infty[\times H$ and a number $\varepsilon > 0$ so that for all $p \in]2, 2 + \varepsilon]$, there exists a **unique** $u_p \in H$ so that (p, u_p) belongs to U and so that u_p is a solution of problem $(\mathcal{P}_{p,k})$.*

Variational formulation of $(\mathcal{P}_{p,k})$

Definition (Action functional)

The action functional $J_{p,k} : H^1_Z \to \mathbb{R}$ is defined by

$$J_{p,k}(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{a}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{\lambda_k}{p}\|u\|^p_{L^p(\mathcal{G})}.$$

Solutions of $(\mathcal{P}_{p,k})$ correspond to critical points of $J_{p,k}$ over H^1_Z.

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One cannot find minimizers of $J_{p,k}$ over H^1_Z. Indeed, if $u \neq 0$, then

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A common strategy to obtain a suitable notion of minimizers is to introduce the *Nehari manifold*, as in Colette's talk.

The Nehari manifold

Definition (Nehari manifold)

The Nehari manifold is defined by

$$\begin{aligned}\mathcal{N}_{p,k}(\mathcal{G}) &:= \left\{u \in H^1_Z(\mathcal{G}) \setminus \{0\} \mid J'_{p,k}(u)[u] = 0\right\} \\ &= \left\{u \in H^1_Z(\mathcal{G}) \setminus \{0\} \mid \|u'\|^2_{L^2(\mathcal{G})} + a\|u\|^2_{L^2(\mathcal{G})} = \lambda_k \|u\|^p_{L^p(\mathcal{G})}\right\}.\end{aligned}$$

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If $u \in \mathcal{N}_{p,k}(\mathcal{G})$, then

$$J_{p,k}(u) = \lambda_k \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|^p_{L^p(\mathcal{G})}.$$

In particular, $J_{p,k}$ is bounded from below on $\mathcal{N}_{p,k}(\mathcal{G})$.

Ground states

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A ground state is a function $u \in \mathcal{N}_{p,k}(\mathcal{G})$ attaining the minimum of $J_{p,k}$ over $\mathcal{N}_{p,k}(\mathcal{G})$.

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Ground states always exist when $\mathcal{G}$ is compact and provide positive solutions to $(\mathcal{P}_{p,k})$.

Nodal ground states

Definition (Nodal Nehari set)

The nodal Nehari set is defined by

$$\mathcal{N}^{\pm}_{p,k}(\mathcal{G}) := \left\{ u \in H^1_Z(\mathcal{G}) \setminus \{0\} \mid u^+ \in \mathcal{N}_{p,k}, u^- \in \mathcal{N}_{p,k} \right\}.$$

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Nodal ground states always exist when $\mathcal{G}$ is compact and provide sign-changing solutions to $(\mathcal{P}_{p,k})$ with two nodal zones.

Variational formulation of the reduced problem on E_k

Definition (Reduced functional)

The reduced functional $J_{*,k} : E_k \to \mathbb{R}$ is defined by

$$J_{*,k}(\psi) := \frac{\lambda_k}{4} \int_{\mathcal{G}} \psi^2 (1 - 2 \ln |\psi|) \, \mathrm{d}x.$$

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- The reduced functional is $\mathcal{C}^2$ on $E_k \cap S$;
- Solutions of the reduced problem correspond to critical points of $J_{*,k}$;
- Nondegenerate solutions of the reduced problem correspond to nondegenerate critical points of $J_{*,k}$.

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Remark: since $\dim E_1 = 1$, $\mathcal{N}_{*,1}(\mathcal{G})$ only contains two elements: a positive one and a negative one.

Uniqueness of positive solutions for $p \approx 2$

Theorem

If $p \approx 2$ is close enough to 2, the positive solution of $(\mathcal{P}_{p,1})$ is unique and is a ground state of the problem.

Main ingredients of the proof.

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- Since $\dim E_1 = 1$, u_* is a nondegenerate critical point of J_*;
- The Lyapunov-Schmidt reduction proves the uniqueness result. □

Convergence of nodal ground states when $p \to 2$

Theorem (Convergence of nodal ground states)

If $(u_{p_n})_n$ is a sequence of nodal ground states of $(\mathcal{P}_{p,k})$ with $p_n \to 2$, then up to a subsequence one has that

$$u_{p_n} \xrightarrow[n\to\infty]{H^2} u_*,$$

where $u_* \in E_2$ minimizes J_* over $\mathcal{N}_{*,2}$.

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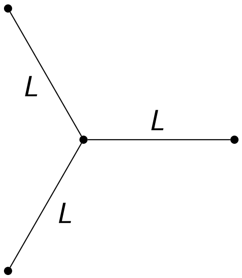
where $u_* \in E_2$ minimizes J_* over $\mathcal{N}_{*,2}$.

If u_* belongs to S (i.e. does not vanish on any edge) and is nondegenerate, one may then obtain uniqueness and symmetry results by using the Lyapunov-Schmidt reduction.

The graph $\mathcal{G}_L$



A compact symmetric 3-star graph with edges of length L.

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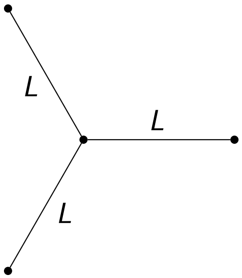

A compact symmetric 3-star graph with edges of length L.

Definition (Symmetric functions on $\mathcal{G}_L$)

A function $u : \mathcal{G}_L \to \mathbb{R}$ is *symmetric* if its restrictions to all edges, viewed as functions $[0, L] \to \mathbb{R}$, are all equal.

Symmetry breaking

Proposition

For any $p > 2$, if L is large enough then the ground state on $\mathcal{G}_L$ is not symmetric.

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A rearrangement argument implies that symmetric functions have a too high action compared to suitable nonsymmetric ones. Since L is large, the level of the *soliton* in $H^1(\mathbb{R})$ plays an important role. □

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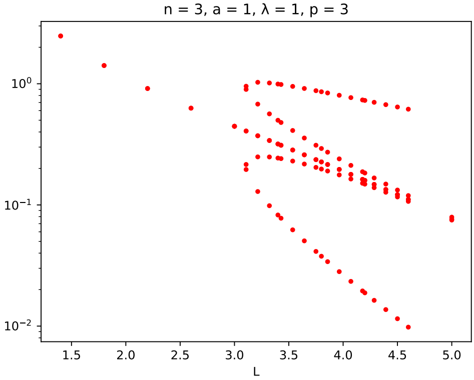
Conclusion

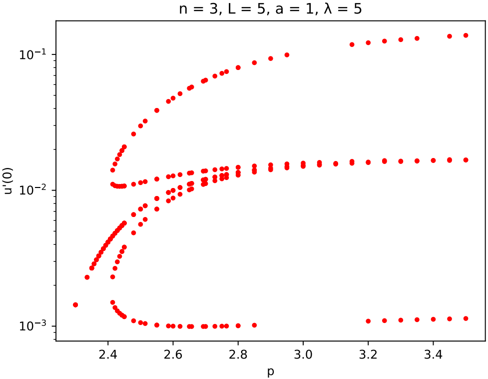
Symmetry breaking occurs!

A bifurcation diagram for positive solutions on $\mathcal{G}_L$: p is fixed, L varies

Vertical axis: possible values of $u_1'(0)$, the derivative of the solution on one given edge of the graph:



Another bifurcation diagram: L is fixed, p varies



What about p close to 2?

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The first step is to identify the second eigenspace E_2.
Identifying functions on $\mathcal{G}_L$ with triples of functions from $[0, L]$ to $\mathbb{R}$, we obtain

$$E_2 = \Big\{(k_1 \sin(x\pi/L), k_2 \sin(x\pi/L), k_3 \sin(x\pi/L)) \mid k_1 + k_2 + k_3 = 0\Big\}.$$

Minimizers points of the reduced functional

Proposition

The set of minimizers of $J_{*,2}$ on $\mathcal{N}_{*,2}$ is

$$S_m := \Big\{(k_m, -k_m, 0), (-k_m, k_m, 0), (k_m, 0, -k_m), (-k_m, 0, k_m), (0, k_m, -k_m), (0, -k_m, k_m)\Big\},$$

with $k_m := \frac{2}{\sqrt{e}}$.

Two asymptotic results

Theorem (2024?)

For any $p > 2$, if L is long enough, then NGS on $\mathcal{G}_L$ vanish on one edge.

Theorem (2024?)

For any $L > 0$, if $p > 2$ is close enough to 2, then NGS on $\mathcal{G}_L$ vanish on one edge.

Summary of what we know about $\mathcal{G}_L$

	GS	NGS
$p \to 2$ (L fixed)	Unique, symmetric	Vanish on one edge
$L \to +\infty$ ($p > 2$ fixed)	Not symmetric	Vanish on one edge

What to do when u_* vanishes on an edge?

In the Lyapunov-Schmidt reduction, we can only deal with eigenfunctions not vanishing on any edge (i.e. $u_* \in S$) *due to regularity reasons*.

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In the Lyapunov-Schmidt reduction, we can only deal with eigenfunctions not vanishing on any edge (i.e. $u_* \in S$) *due to regularity reasons*.

Question

Assuming $\dim E_k = 1$, *can we perform a Lyapunov-Schmidt reduction starting from an eigenfunction vanishing on one edge of a graph? Can we obtain existence and uniqueness results of solutions of the nonlinear problem close to* u_**?*

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When studying properties of ground states and nodal ground states on compact symmetric stars, we use the fact that *when the edges are long, (nodal) ground states look like portions of solitons.*

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Can the asymptotic arguments be dropped in order to identify more precise thresholds of:

- *symmetry vs asymmetry;*
- *solutions vanishing on edge vs solutions in S?*

Thanks for your attention!

References for $p \to 2$

The $p \to 2$ analysis was largely based on

M. Grossi, ⚠On the shape of solutions of an asymptotically linear problem ⚠, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), Vol. VIII (2009), 429–449.

D. Bonheure, V. Bouchez, C. Grumiau, J. Van Schaftingen Asymptotics and symmetries of least energy nodal solutions of Lane-Emden problems with slow growth, Communications in Contemporary Mathematics, Vol. 10, No. 4 (2008) 609–631.

C. Grumiau, Symmetries of solutions for nonlinear Schrödinger equations: Numerical and Theoretical approaches, PhD Thesis (UMONS, 2010-2011).

A more precise result on NGS vanishing on edges

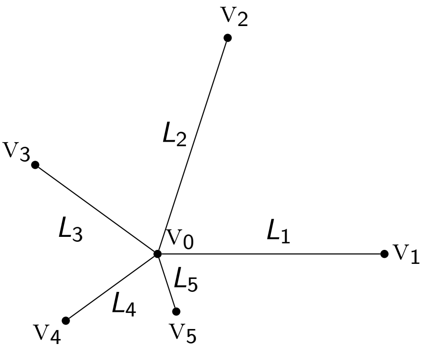


Figure: A graph under study: $L_1 = L_2 \geq L_3 \geq L_4 \geq L_5$.

For such a graph, if $a > 0$ and $p > 2$ are fixed, there exists $\bar{L} > 0$ so that if $L_1 \geq \bar{L}$ then NGS vanish on all edges except two of length L_1.

Maximizers of $J_{*,2}$

Theorem

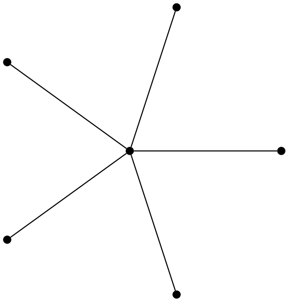
The set of maximizers of $J_{*,2}$ is

$$S_M := \Big\{(k_M, k_M, -2k_M), (-k_M, -k_M, 2k_M), (k_M, -2k_M, k_M), (-k_M, 2k_M, -k_M), (-2k_M, k_M, k_M), (2k_M, -k_M, -k_M)\Big\},$$

with $k_M := \frac{\sqrt[3]{2}}{\sqrt{e}}$.

They correspond to solutions of the nonlinear problem that can be found by shooting methods or variationally.

The second eigenspace may have high dimension



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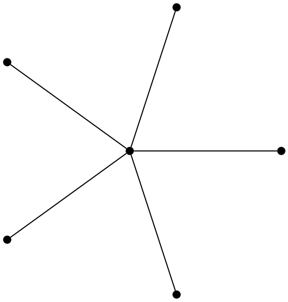


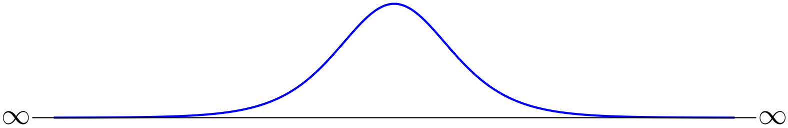
Example

Computations on the blackboard!

An important $H^1(\mathbb{R})$ solution: the soliton φ_p $(a > 0)$

For every $p > 2$, we consider the *soliton* φ_p, the unique positive and even solution to

$$-u'' + au = \lambda_1(\mathcal{G}_L)|u|^{p-2}u.$$



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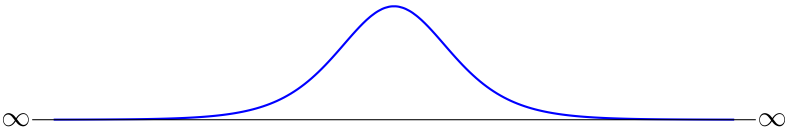

The level of φ_p is important to study solutions on $\mathcal{G}_L$ when L is large:

$$s_p := J_{p,1}(\varphi_p).$$

More details on the proof of symmetry breaking

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A rearrangement argument implies that the action of any symmetric function in $\mathcal{N}_p$ is larger than $\frac{3}{2} s_p$.

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Proof.

A rearrangement argument implies that the action of any symmetric function in $\mathcal{N}_p$ is larger than $\frac{3}{2}s_p$. Moreover, $\inf_{\mathcal{N}_p} J_p$ gets close to s_p when L is large. □

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Another asymptotic regime: $p \to +\infty$

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About NGS ($\Omega \subseteq \mathbb{R}^2$):

M. Grossi, C. Grumiau, F. Pacella, Lane–Emden problems: Asymptotic behavior of low energy nodal solutions, Ann. I. H. Poincaré – AN 30 (2013) 121–140.

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